# Game Theory

## Lecture 1: Expected utility – economic decision under uncertainty

Frieder Neunhoeffer

ISEG Lisbon School of Economics & Management
Universidade de Lisboa

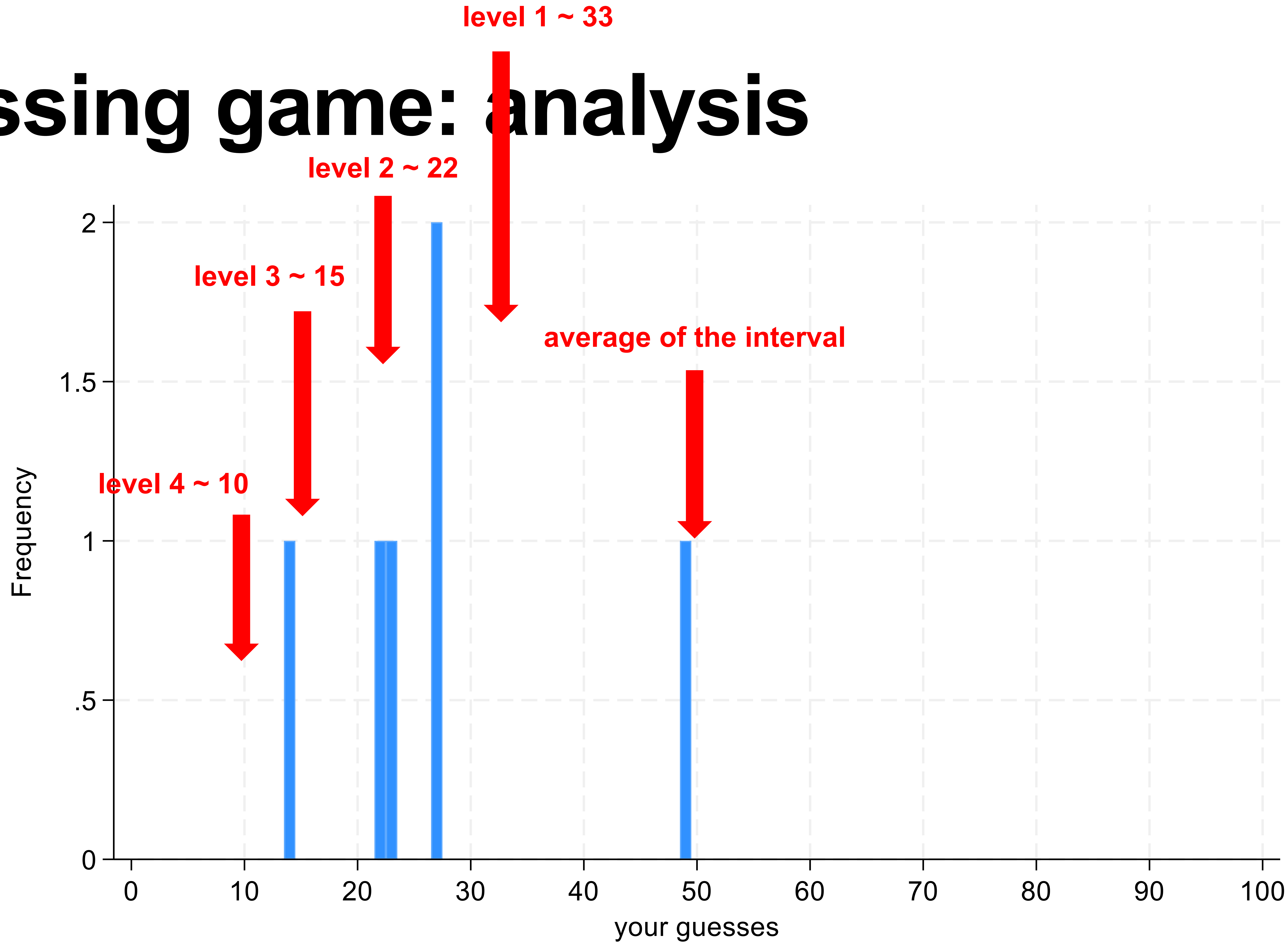
U LISBOA | UNIVERSIDADE DE LISBOA

# Last lecture: guessing game

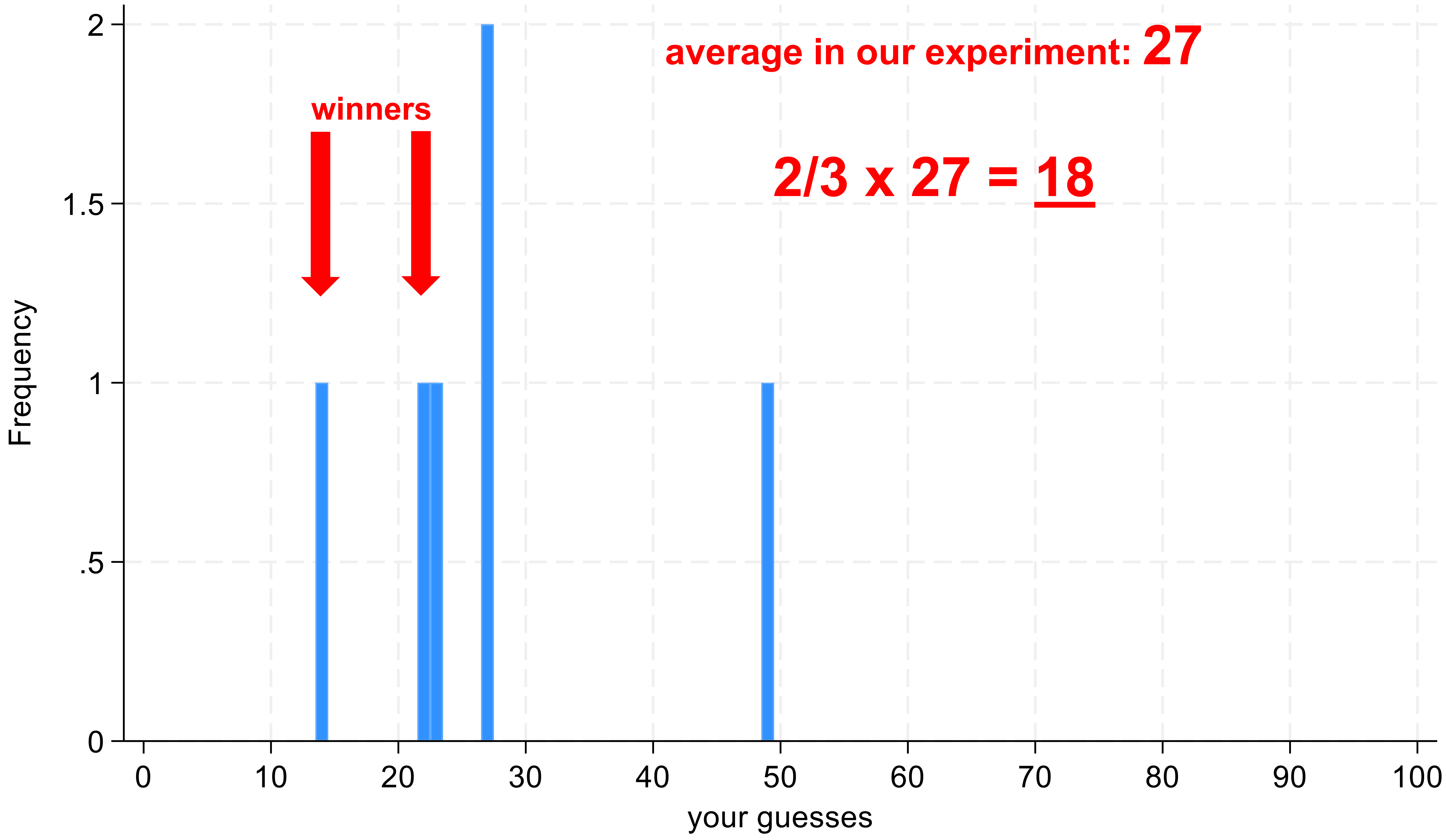
and the winner is…

- Guess an integer from the interval 0 to 100.
- The winner is whose guess is closest to 2/3 times the average of all guesses.

# Guessing game: analysis



# Guessing game: analysis



# What is a “game” in Economics?

Situation of conflict of interests among a finite number of participants (i.e., players). Hence, the definition of a static game is the following:

- Each **player** selects one **strategy** among a set (≥ 2) of strategies.
- A strategy means selecting a certain **action** (e.g., bus or car, certain number in guessing game)
- Strategy choices take place **simultaneously**, excluding that players can observe a rival’s choice and react to it.
- The combination of strategies generates an **outcome**, which is evaluated by each player, according to her **preferences**.
- The problem that each player faces is how to use her **partial influence** on the outcome in order to benefit as much as possible → *homo economicus*
- Each player knows that her rivals behave in the same way as her → **common knowledge**

# Probabilities and lotteries

- There are two kinds of actions
  1. certain actions generate a single outcome
  2. uncertain actions (lotteries) can yield different outcomes

- Let us assume a **lottery**, where:
  - $x_i$ is the amount of money associated with outcome, or *state*, $i = 1,2,\dots,n$
  - $p_i$ is the probability, or *relative likelihood*, that outcome $i$ occurs

- Probabilities have two properties:
  1. $p_i \geq 0$: probabilities are non-negative
  2. $\sum_{i=1}^{n} p_i = 1$: probabilities need to sum to 1 (i.e., 100%)

**Meaning of 1 and 2:** If a lottery is performed, there should arise *one and only one* outcome. Outcomes are mutually exclusive and cover all the possibilities.

# Expected value, and variance

- The *average* or *expected* value of the lottery is given by:

$$E\{x\} = \bar{x} = \sum_{i=1}^{n} p_i x_i$$

- <u>Question:</u> Toss a fair coin. What is the expected value of each lottery?

  A. Lottery pays 200€ if heads and 0€ if tails
  B. Lottery pays 50€ if heads and 150€ if tails

- The degree or risk associated with the lottery is given by the dispersion of outcomes around the mean, or *variance*:

$$var\{x\} = \sum_{i=1}^{n} p_i (x_i - \bar{x})^2$$

- <u>Question:</u> Which lottery is riskier?

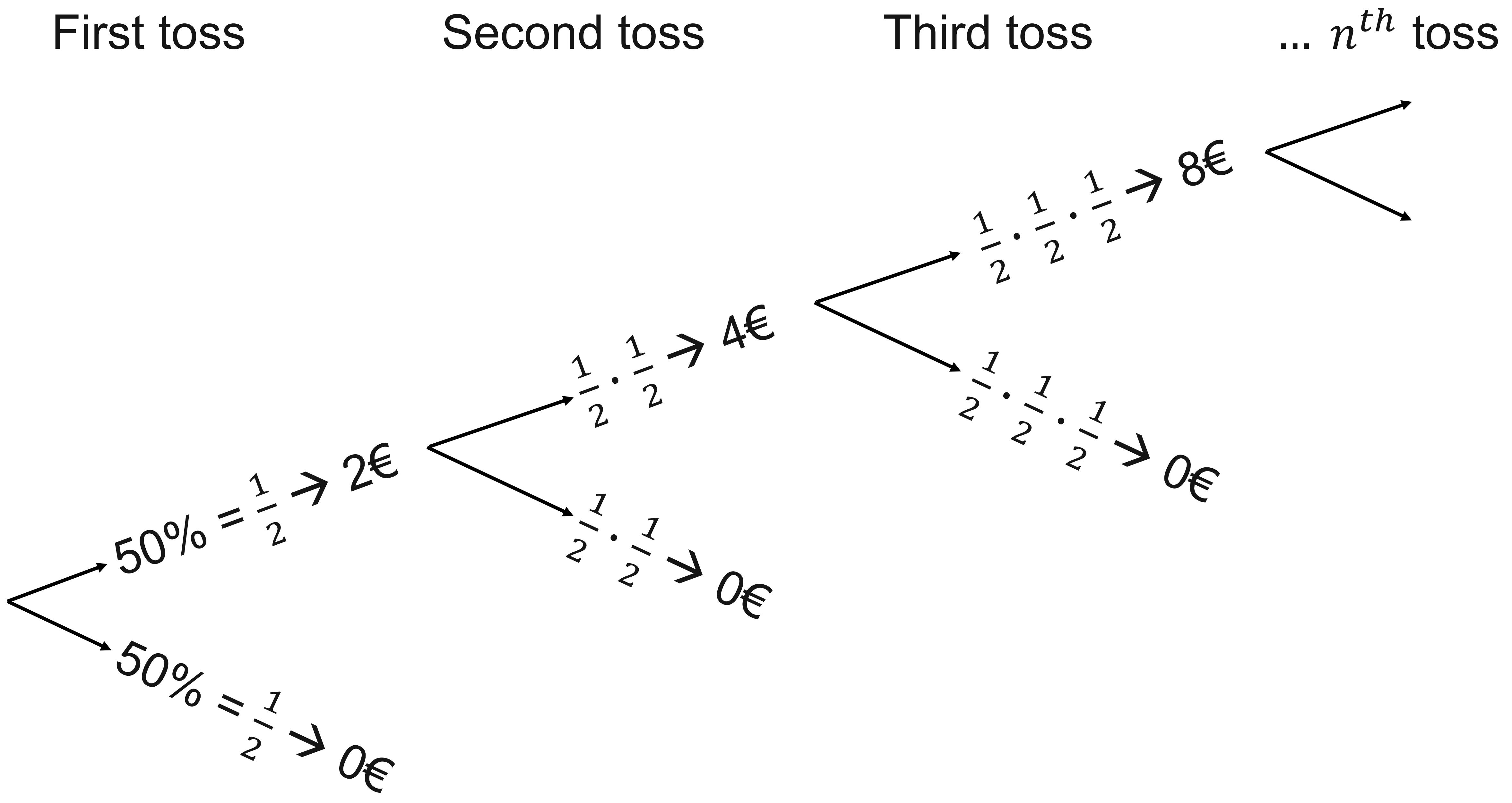
# St. Petersburg paradox

Daniel Bernoulli (1738)

- How much would **you** pay to play?
- What is the expected value (EV)?

1 x → 2 €
2 x → 4 €
3 x → 8 €
n x → $2^n$€

End of the game



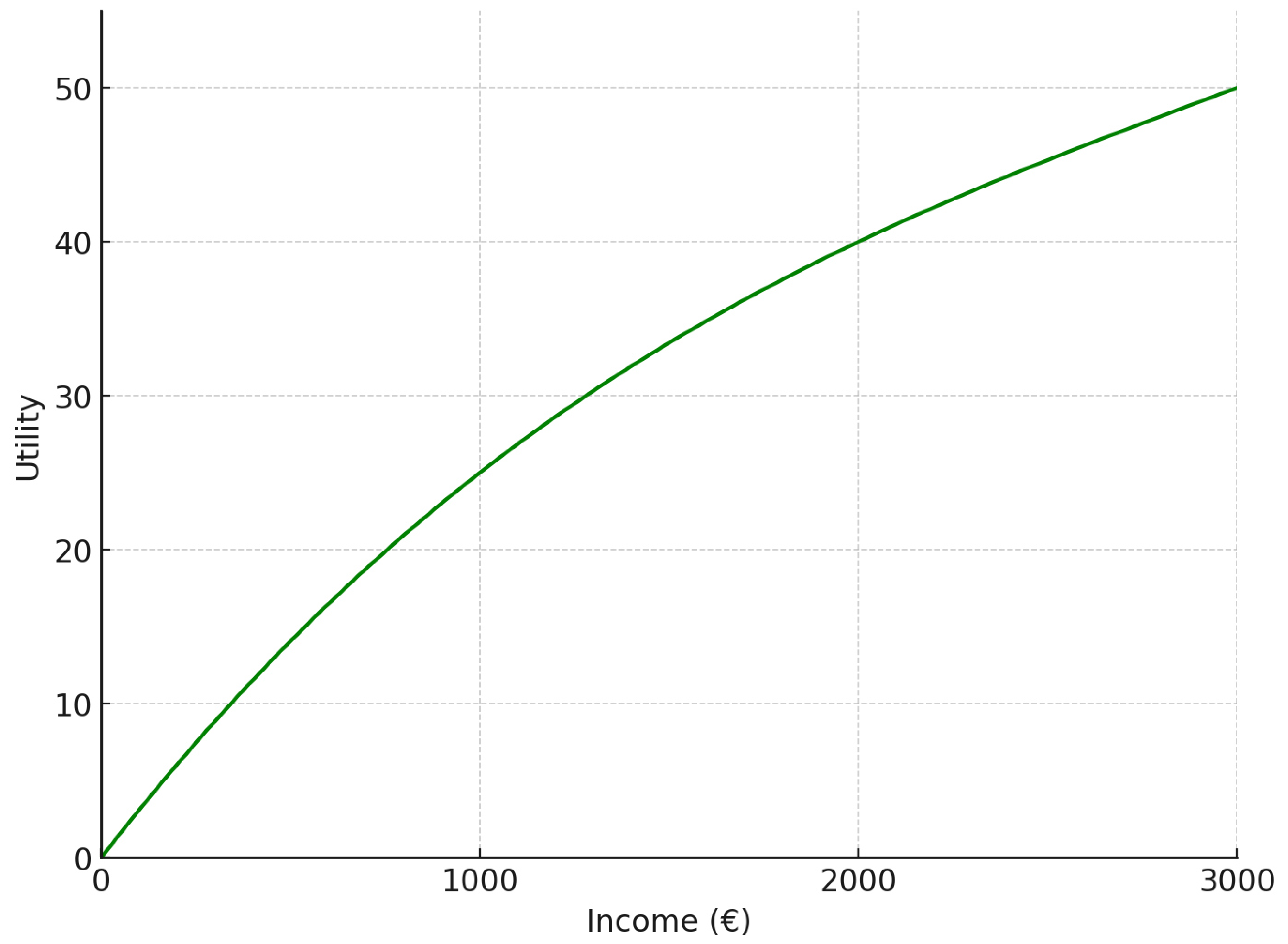

- $EV = \frac{1}{2} \cdot 2€ + \frac{1}{4} \cdot 4€ + \frac{1}{8} \cdot 8€ + \cdots + \frac{1}{2^n} \cdot 2^n€$
  $= 1€ + 1€ + 1€ + \cdots + 1€ = \infty$

# St. Petersburg paradox

Daniel Bernoulli (1738)

Classical solution: *utility function*

→ diminishing marginal utility of money



# Expected utility

- An alternative standard is **expected utility theory (EUT)** introduced by *von Neumann and Morgenstern (1947).*
- Idea: assign to each possible outcome $x_i$ an utility number $U(x_i)$, such that the preference for the lottery is fully characterized by the expected utility $E\{U\}$ :

$$E\{U\} = \sum_{i=1}^{n} p_i U(x_i)$$

- For two lotteries $L$ and $L'$ that share the same set of possible outcomes $(x_1, x_2, \dots, x_n)$, assigning them different probabilities $(p_1, p_2, \dots, p_n), (p'_1, p'_2, \dots, p'_n)$, $L$ is preferred to $L'$ iff:

$$\sum_{i=1}^{n} p_i U(x_i) > \sum_{i=1}^{n} p'_i U(x_i)$$

- The ranking between the two lotteries is determined by the order of expected utilities.
- The rational consumer chooses between risky actions as if he maximized the EU.

# Expected utility axioms by vNM

- Four foundational and necessary principles to construct a utility function in order to represent preferences over uncertain outcomes.

1. *Completeness:*

   - A decision-maker can always compare two outcomes $x_1$ and $x_2$
   - For any two outcomes $x_1$ and $x_2$ , one and only one of the following is true:

$$x_1 \succ x_2\colon\ x_1\ is\ strictly\ preferred\ to\ x_2,$$
$$x_2 \succ x_1\colon\ x_2\ is\ strictly\ preferred\ to\ x_1,$$
$$x_1 \sim x_2\colon\ x_1\ and\ x_j\ are\ eqally\ preferred.$$

   - **Interpretation:** Preferences are well-defined and complete.

# Expected utility axioms by vNM

2. *Transitivity:*
   - If $x_1 \succ x_2$ and $x_2 \succ x_3$, then $x_1 \succ x_3$.
   - If $x_1 \sim x_2$ and $x_2 \sim x_3$, then $x_1 \sim x_3$.
   - **Interpretation:** Preferences are logically consistent.

3. *Independence:*
   - If $x_1 \succ x_2$, then for any probability $p$ (where $0 < p < 1$) and a third outcome $x_3$:
   $$px_1 + (1-p)x_3 \succ px_2 + (1-p)x_3,$$
   - **Interpretation:** Decisions about $x_1$ and $x_2$ should not depend on irrelevant alternatives.

# Expected utility axioms by vNM

- Let us assume that the n possible outcomes of a lottery are numbered so that $x_1$ is the least preferred outcome and $x_n$ is the most preferred one.

$$x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n$$

*4. <u>Continuity:</u>*

- For any outcome $x_i$ between $x_1$ and $x_n$, there is a probability $p_i$ such that she is indifferent between getting $x_i$ with certainty or to play a lottery where she obtains $x_n$ with probability $p_i$ and $x_1$ with probability $(1 - p_i)$. We say $x_i$ is the certainty equivalent of lottery $\tilde{x}_i$, where

$$\tilde{x}_i \equiv \begin{pmatrix} x_n \; with \; probability \; p_i, \\ and \; x_1 \; with \; probability \; (1 - p_i) \end{pmatrix}$$

- **Interpretation:** There are no "jumps" in preferences; they change smoothly.

# Expected utility axioms by vNM

- **Monotonicity** is not an explicit vNM axiom but rather a derived principle that naturally arises from the Independence and Continuity axioms. In some cases, it may be treated as an additional assumption to emphasize probability weighting in preferences.
- *Monotonicity:*
  - If two lotteries with the same two outcomes differ only in the probabilities assigned to each outcome, the lottery that gives the highest probability to the better alternative is preferred to the other lottery, i.e.:

$$px_2 + (1-p)x_1 \succ p'x_2 + (1-p')x_1,$$

  - iff $p > p'$ and $x_2 \succ x_1$

# Constructing the von Neumann–Morgenstern utility index

- The possible outcomes are ordered as $x_1, x_2, \ldots, x_n$, where $x_1 \prec x_2, \ldots, x_{n-1} \prec x_n$.
- A utility value 0 is assigned to the least preferred outcome $x_1$. A utility number 1 is assigned to the most preferred outcome $x_n$.
- To any other outcome $x_i$, a utility number equal to $p_i$ is assigned such that $x_i$ is the certainty equivalent of a lottery giving $x_n$ with probability $p_i$ and $x_1$, with probability $(1 - p_i)$.

$$U(x_1) \equiv 0$$
$$U(x_n) \equiv 1$$
$$U(x_i) \equiv p_i$$

- These assignments are consistent with the expected utility rule because:

$$U(x_i) = p_i U(x_n) + (1 - p_i) U(x_1) = p_i + 0 = p_i$$

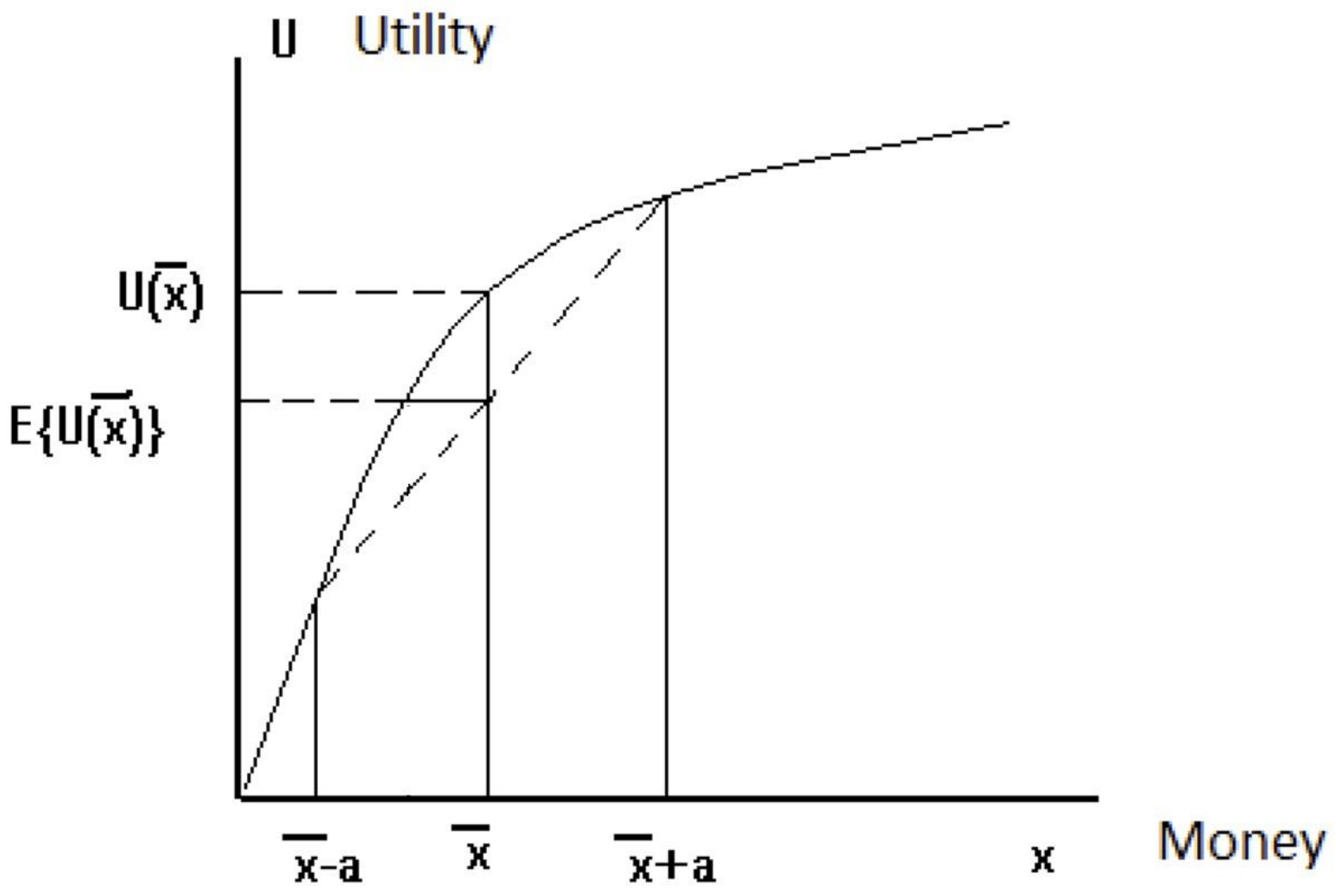
- The von Neumann-Morgenstern index is constant up to a positive linear transformation.

# Consumer's attitude towards risk

- A lottery $\bar{x}$ gives two outcomes, $\bar{x} - a$ and $\bar{x} + a$ with the same probability $\frac{1}{2}$.
- Let $E(\bar{x}) = \bar{x}$ be the expected value (EV) of the lottery.
- A consumer is *risk-averse* if she always prefers the EV with certainty to a lottery with the same EV but with a positive variance.
- A consumer is *risk-neutral* if she is indifferent between the certain EV and the lottery.
- A consumer is *risk-loving* or *risk-seeking* if she strictly prefers the lottery.
- $U(E(\bar{x}))$ is the utility of the certain EV of the lottery.
- $E\{U(\bar{x})\} = U(\bar{x} - a)\frac{1}{2} + U(\bar{x} + a)\frac{1}{2}$ is the expected utility of the lottery, expressed by the medium point in the line that unites points $\left(\bar{x} - a, U(\bar{x} - a)\right)$ and $\left(\bar{x} + a, U(\bar{x} + a)\right)$ in curve $U$.

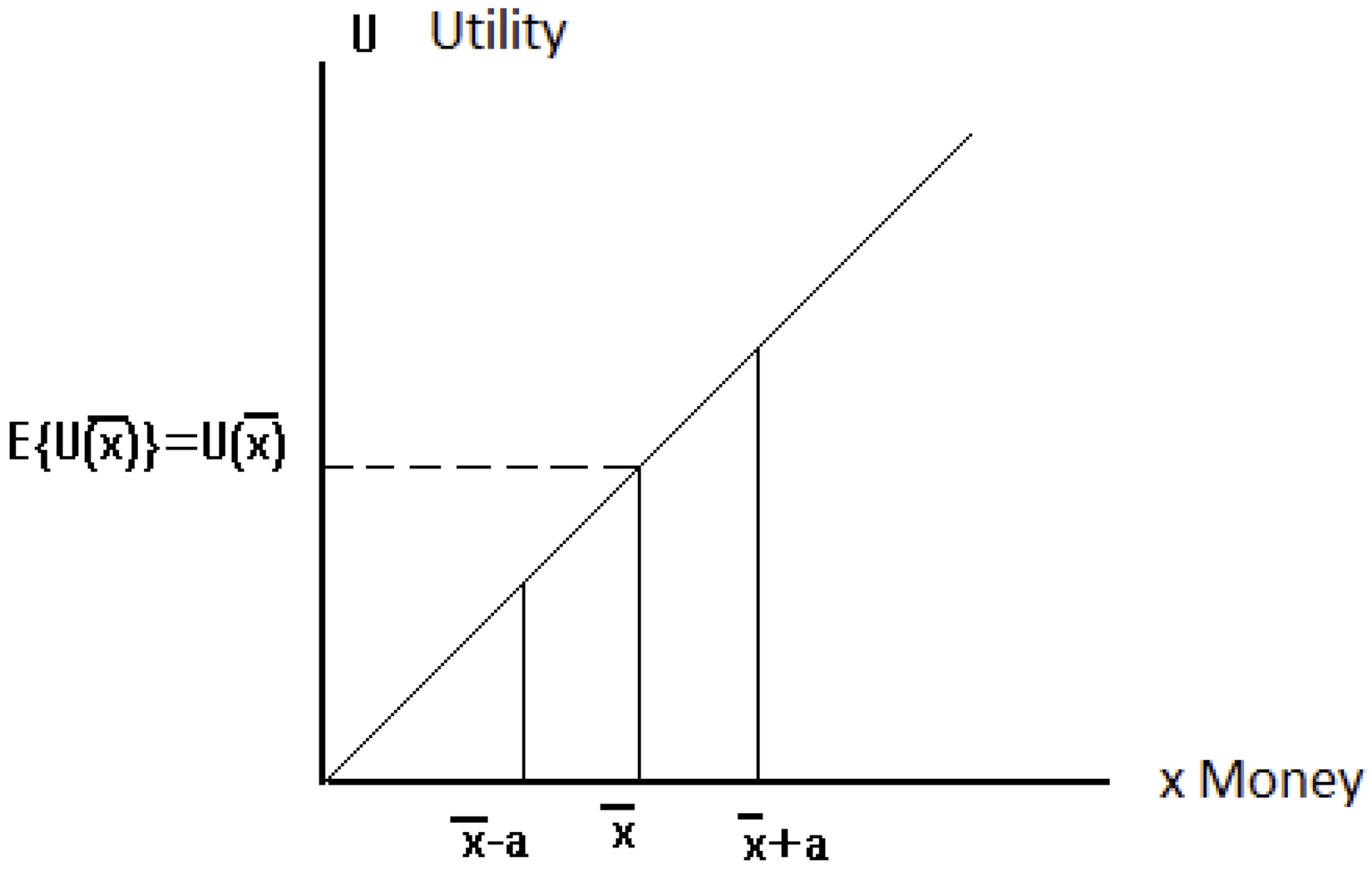
# Utility function of a risk-averse individual

- $U(\bar{x}) > E\{U(\bar{x})\}$ if the income utility curve is concave. There is risk aversion.



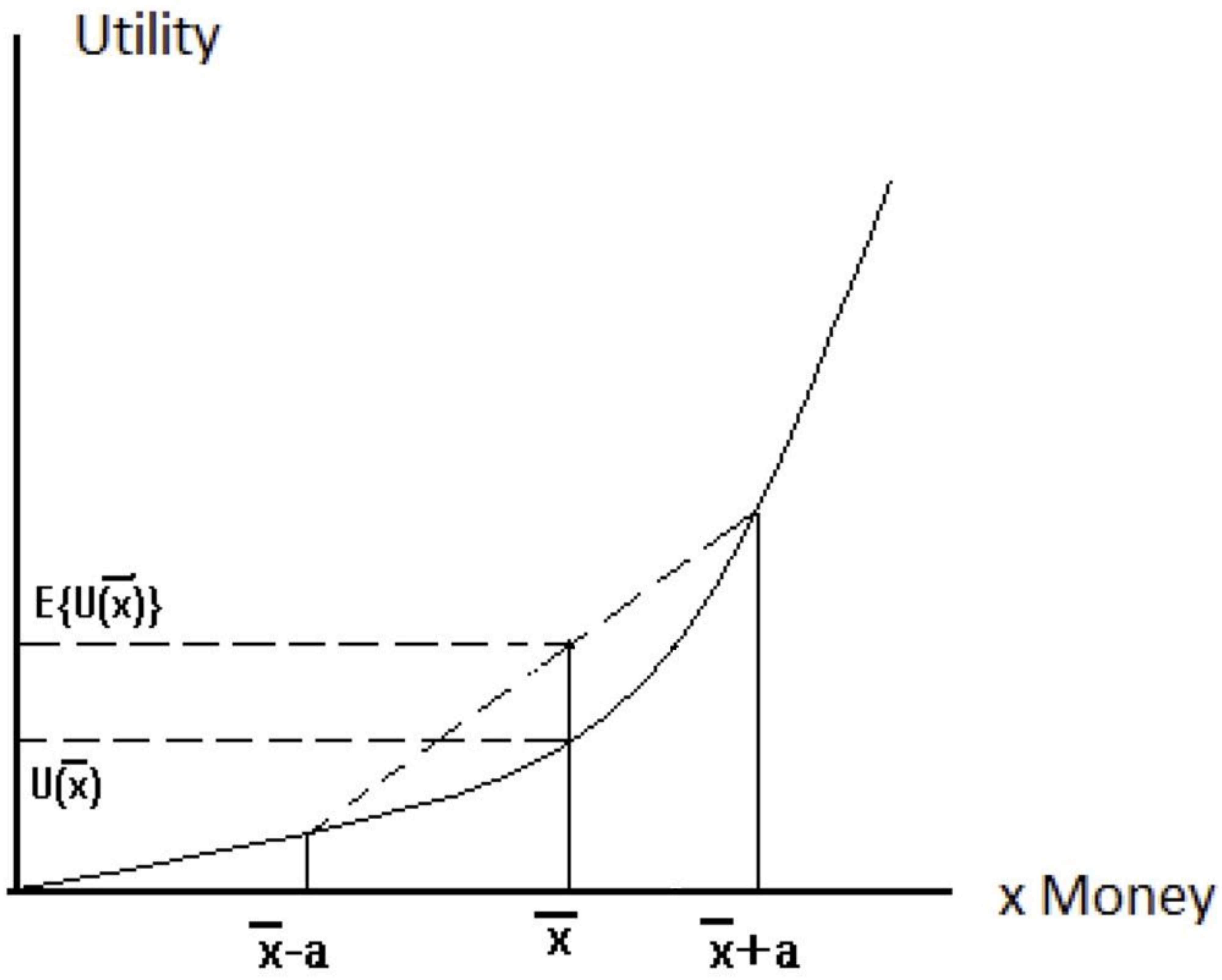
# Utility function of a risk-neutral individual

- $U(\bar{x}) = E\{U(\bar{x})\}$ if the income utility curve is linear. There is risk neutrality.



# Utility function of a risk-seeking individual

- $U(\bar{x}) < E\{U(\bar{x})\}$ if the income utility curve is convex. There is risk seeking.



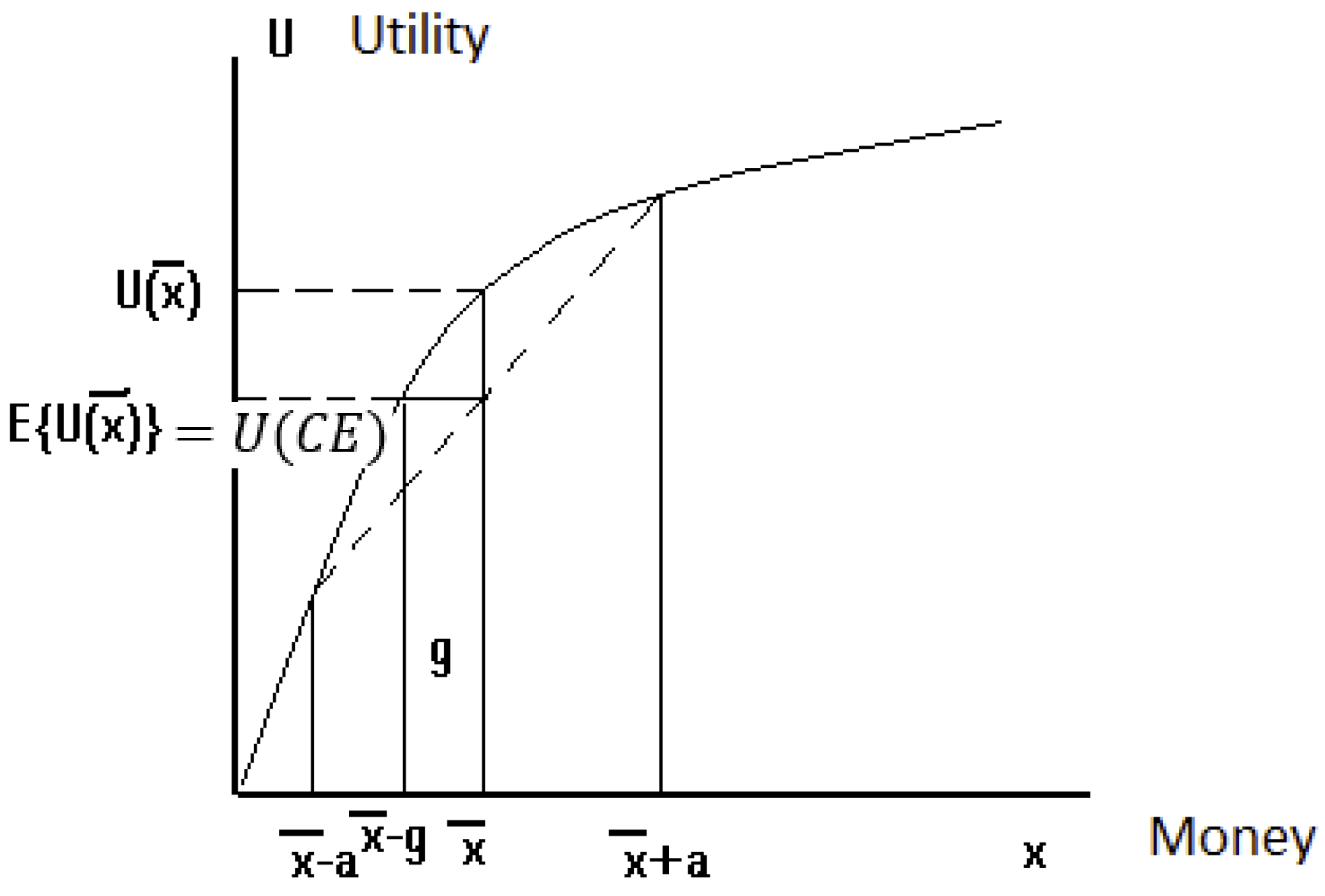
# Risk premium

- We can also express risk aversion in another way. We define a sum of money $g$, such that $\bar{x} - g$ is the certainty equivalent (CE) of the lottery. Hence, we have

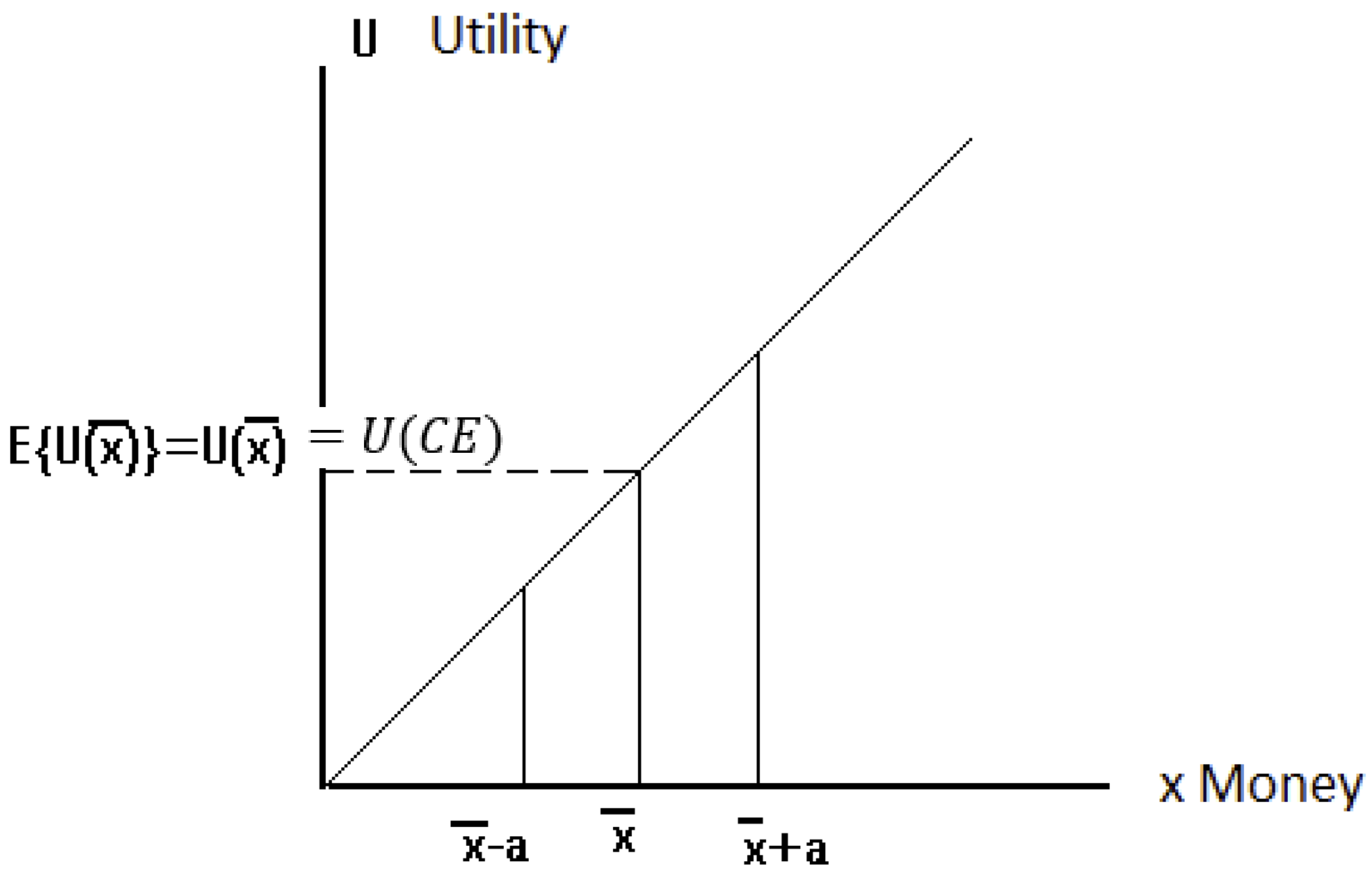
$$U(\bar{x} - g) = U(CE) = E\{U(\bar{x})\}$$

- $g$ is called a *risk premium*. It is
  - positive, in the case of risk aversion,
  - zero in the case of risk neutrality,
  - negative in the case of risk loving.
- The lottery represents the situation without insurance.
- The certainty equivalent (CE) stands for the situation with full insurance.
- $g$ represents the price (or "premium") of an insurance with complete covering of loss.

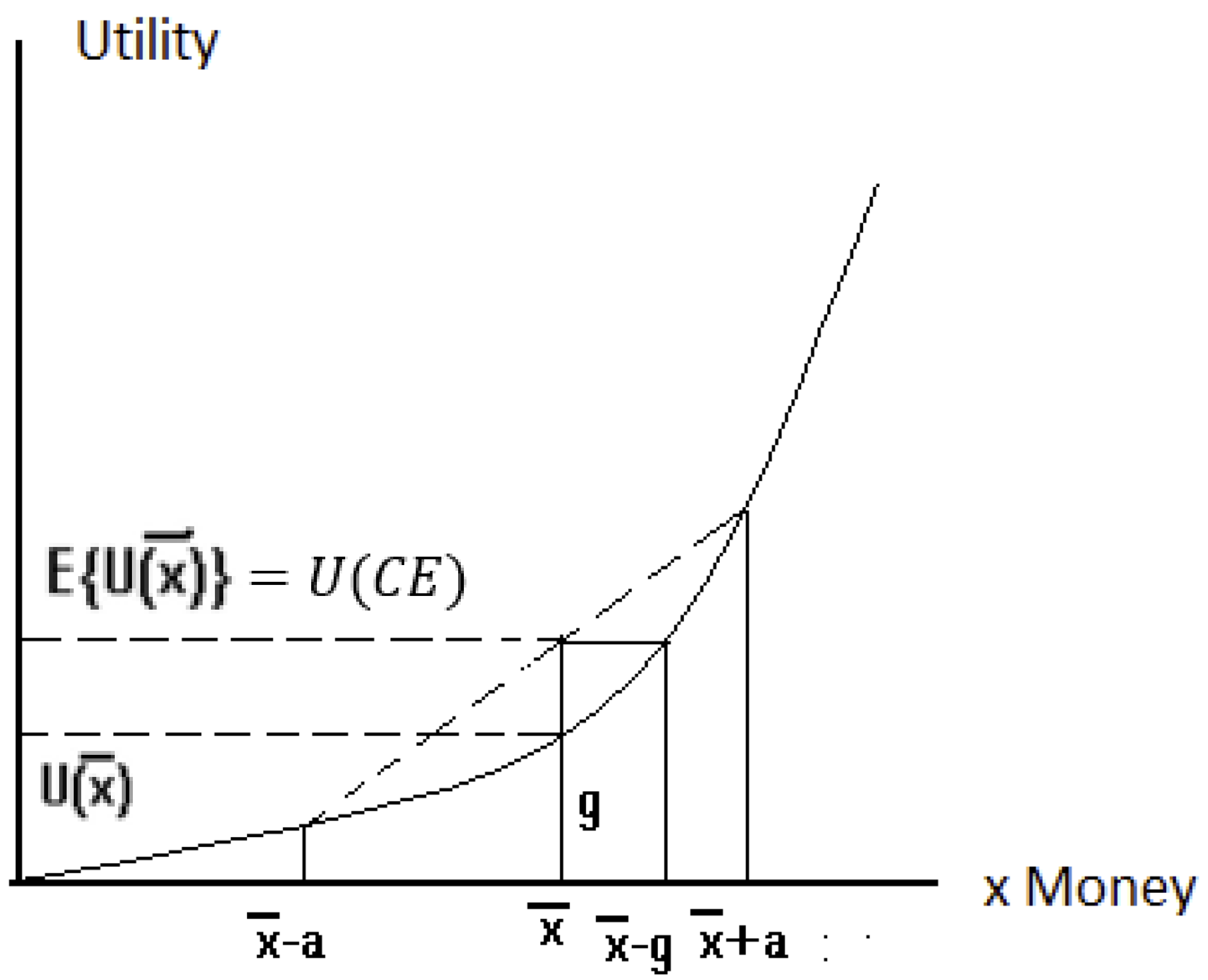
# Risk premium




Risk aversion




Risk neutrality




Risk seeking

# Risk aversion measures

- Let us assume that an individual is risk averse, so that her utility function is increasing and concave. The *Arrow-Pratt* measure $r_u \geq 0$ of absolute risk aversion is defined by:

$$r_u = -\frac{u''(x)}{u'(x)}$$

- If $r_u = 0$, then $u(x)$ is linear and the individual is risk neutral.
- An individual with utility function $u$ is said to be more risk averse than an agent with utility function $v$ if $r_u \geq r_v$ for all values of $x$

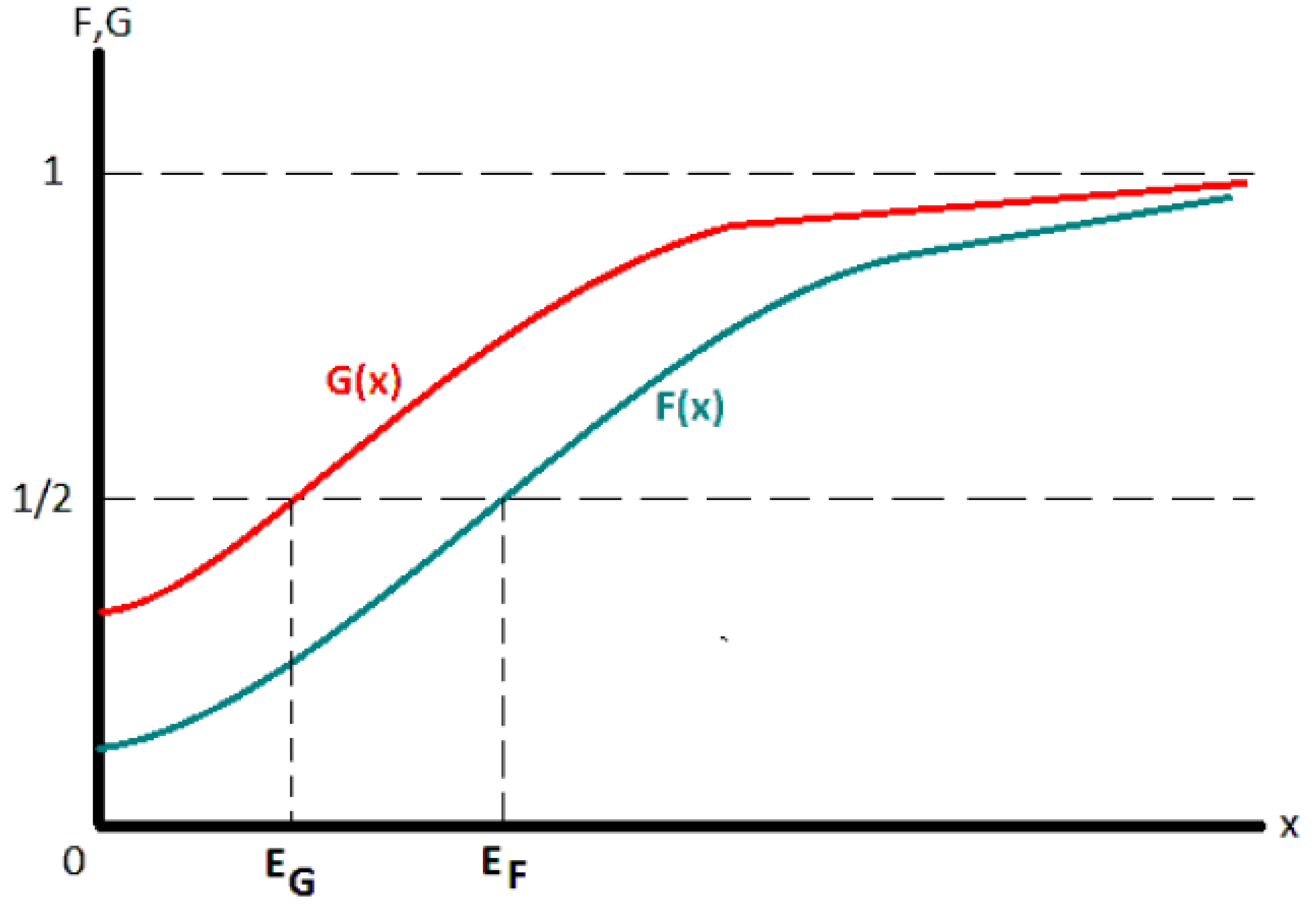
# First order stochastic dominance

- Assume that two random variables, $W$ and $Y$, have cumulative distribution functions $F$ and $G$, respectively. Then, we say that $W$ is stochastically larger than $Y$, iff

$$G(x) - F(x) \geq 0 \text{ for all } x$$

- It is easy to show that the expected value of $W$ is higher than the expected value of $Y$.

- If we assume that the distribution functions are symmetric, then the mean of each distribution is coincident with the median.
(the value taken by the distribution for $x = \frac{1}{2}$)



# Second order stochastic dominance

- We say that random variable $W$ is stochastically "less risky" than random variable $Y$ if

$$\int_0^x [G(s) - F(s)]\, ds \geq 0 \text{ for all } x$$

- Its meaning becomes clearer in the case where the exp-ected values of $W$ ($\rightarrow F(x)$) and $Y$ ($\rightarrow G(x)$) are equal, i.e., $G$ represents a "mean preserving spread" in relation to $F$.

- The overall area comprised between curves $G(x)$ and $F(x)$ is positive for any finite value of $x$, so that the condition of second order stochastic dominance is met.

- Moreover, the degree of concentration of values around the mean is higher for distribution $F$ than for distribution $G$, so that $W$ is less risky than $Y$.